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# UPPER SEMICONTINUOUS GLOBAL ATTRACTORS FOR VISCOUS FLOW

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## ABSTRACT

A particular form of regularization of the Navier-Stokes equations is studied. It has been shown that as the regularization parameter goes to zero, the global attractor for the regularized system converges to the global attractor for the conventional Navier-Stokes system.

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# 1 Introduction

Two of the profound open problems in the theory of three dimensional viscous flow are the unique solvability theorem for all time and the existence theorem for the global attractor. We have shown in our earlier studies [10, 9, 11, 13] that certain regularizations of the Navier-Stokes equations are uniquely solvable (up to dimension six) and can be characterized by compact global attractors. A natural question then is to investigate the possibility of establishing such results for the conventional Navier-Stokes equations by a limit process. In this paper we will accomplish this for the two dimensional case. We prove in particular that the compact global attractor $\Lambda_\epsilon$ for the regularized system converges to the compact global attractor $\Lambda$ of the conventional system as $\epsilon \to 0$.

The outline of the paper is as follows. In §2 we present the relevant mathematical framework for the paper. In §3 we establish several uniform estimates for the generalized solutions of the regularized system. These estimates hold independent of the size of the regularization parameter and remain valid when this parameter goes to zero. In §4 we show that the solution of the regularized system converges to the solution of the conventional system as the regularization parameter goes to zero. The next two sections deal with convergence of attractors. In §5 we establish the convergence of the simple case of time periodic solutions. Then in §6 we prove the central result of this paper establishing the upper semicontinuity of the compact global attractor $\Lambda_\epsilon$ at $\epsilon = 0$.

The question of the convergence of $\Lambda_\epsilon$ to $\Lambda$ is thus completely answered for the two dimensional case. Our future investigations will be concerned with the corresponding problem for the three dimensional case. We have already reported partial results in this direction in [10, 9] where it was shown that the generalized solutions of the three dimensional regularized system converges to the solution of the conventional system under certain conditions. The methods validated in this paper certainly give us guidelines to elaborate the three dimensional case. We remark here that the use of the artificial viscosity method to establish solvability has also been successful in other branches of partial differential equations. A well

known example of such a result is the viscosity solution method for the Hamilton-Jacobi equations [2].

## 2 Notations and Preliminaries

We regularize the Navier-Stokes system by adding a fourth order artificial viscosity term (Laplacian square) to the conventional system. In this paper we will restrict ourselves to periodic boundary conditions. A thorough study of the regularized system with this and other types of boundary conditions was carried out in [10, 9, 11].

Let us consider the velocity field $\boldsymbol{u} = (u_1, u_2)$ and pressure field $p$ with space periodic condition in $\boldsymbol{R}^2$ such that for $\Omega = (0, L) \times (0, L)$, we have

$$\frac{\partial \boldsymbol{u}}{\partial t} + \epsilon\Delta^2\boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \text{in} \quad \Omega \times (0, \infty), \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \text{in} \quad \Omega \times (0, \infty), \tag{2}$$

$$\boldsymbol{u}(\boldsymbol{x} + Le_i, t) = \boldsymbol{u}(\boldsymbol{x}, t) \quad i = 1, 2, \qquad \forall \quad t \in (0, \infty), \tag{3}$$

$$p(\boldsymbol{x} + Le_i, t) = p(\boldsymbol{x}, t) \quad i = 1, 2, \qquad \forall \quad t \in (0, \infty), \tag{4}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \text{in} \quad \Omega. \tag{5}$$

Here $\nu > 0$ is the coefficient of the kinematic viscosity of the fluid and $\epsilon > 0$ is the artificial dissipation parameter. $\boldsymbol{f}$ and $\boldsymbol{u}_0$ are prescribed vectorfields. Notice that the conventional Navier-Stokes equations correspond to the value of the regularized parameter $\epsilon = 0$.

We denote by $H^m(\Omega)$, the Sobolev space of $L$-periodic generalized functions (condition (3)) which have up to order $m$ square integrable distributional derivatives. These spaces are endowed with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_m = \sum_{|\alpha| \leq m} (D^{\alpha}\boldsymbol{u}, D^{\alpha}\boldsymbol{v})_{L^2(\Omega)}$$

and the norm

$$\|\boldsymbol{u}\|_m = \Big(\sum_{|\alpha| \leq m} \|D^{\alpha}\boldsymbol{u}\|^2_{L^2(\Omega)}\Big)^{1/2}.$$

Suppose we expand $\boldsymbol{u} \in H^m(\Omega)$ by the Fourier series

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{k\in Z^2} c_k e^{2i\pi k\cdot \boldsymbol{x}/L},$$

then the Sobolev space $\boldsymbol{u} \in H^m(\Omega)$ can be characterized by

$$H^m(\Omega) = \{\boldsymbol{u} \mid \bar{c}_k = c_{-k}, \ \sum_{k\in Z^2} |k|^{2m}|c_k|^2 < \infty\}.$$

We denote by $\dot{H}^m(\Omega)$ the subspace of $H^m(\Omega)$ with zero average:

$$\dot{H}^m(\Omega) = \{\boldsymbol{u} \in H^m(\Omega)\,; \ \int_\Omega \boldsymbol{u}(\boldsymbol{x})d\boldsymbol{x} = 0\}.$$

For $m = 0$, we have $\dot{H}^0(\Omega) = \dot{L}^2(\Omega)$.

We introduce the following solenoidal subspaces which are important to our analysis:

$$H = \{\boldsymbol{u} \in \dot{L}^2(\Omega), \ \mathrm{div}\boldsymbol{u} = 0, \ \boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_i} = -\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_{i+2}}, \ i = 1,2\};$$

$$\hat{V} = \{\boldsymbol{u} \in \dot{H}^1(\Omega), \ \mathrm{div}\boldsymbol{u} = 0, \ \gamma_0\boldsymbol{u}|_{\Gamma_i} = \gamma_0\boldsymbol{u}|_{\Gamma_{i+2}}, \ i = 1,2\}.$$

$$V = \{\boldsymbol{u} \in \dot{H}^2(\Omega), \ \mathrm{div}\boldsymbol{u} = 0, \ \gamma_0\boldsymbol{u}|_{\Gamma_i} = \gamma_0\boldsymbol{u}|_{\Gamma_{i+2}}, \gamma_1\boldsymbol{u}|_{\Gamma_i} = -\gamma_1\boldsymbol{u}|_{\Gamma_{i+2}}, \ i = 1,2\}.$$

Here the faces of $\Omega$ are numbered as

$$\Gamma_i = \partial\Omega \cap \{x_i = 0\} \quad \text{and} \quad \Gamma_{i+2} = \partial\Omega \cap \{x_i = L\}, \ i = 1,2.$$

Here $\gamma_0, \gamma_1$ are the trace operators and $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$.

The space $H$ is endowed with the inner product $(\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)}$ and norm $|\boldsymbol{u}| = (\boldsymbol{u},\boldsymbol{u})^{1/2}_{L^2(\Omega)}$ (or $\|\cdot\|_0$). It can be shown that the norm induced by $\dot{H}^1(\Omega)$ and the norm $\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}$ are equivalent in $\hat{V}$. Similarly, in $V$ the norm induced by $\dot{H}^2(\Omega)$ is equivalent to the norm $\|\Delta\boldsymbol{u}\|_{L^2(\Omega)}$. We then denote $\|\boldsymbol{u}\| = \|\Delta\boldsymbol{u}\|_{L^2(\Omega)} = (\boldsymbol{u},\boldsymbol{u})_V^{1/2}$ as norm in $V$ derived from the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V = (\Delta\boldsymbol{u},\Delta\boldsymbol{v}).$$

Let $\hat{V}'$, $V'$ denote the dual spaces of $\hat{V}$ and $V$ respectively.

The Stokes operator $A_1$ can be characterized explicitly using Fourier series. We write the Stokes problem,

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla q = \boldsymbol{g}, & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0, & \text{in } \Omega. \end{cases}$$

as

$$A_1 \boldsymbol{u} = \boldsymbol{g}, \qquad \text{with} \qquad D(A_1) = \{\boldsymbol{u} \in H,\ A_1 \boldsymbol{u} \in H\}.$$

We have in fact,

$$\boldsymbol{u} \in D(A_1) = \dot{H}^2(\Omega) \cap H = V. \tag{6}$$

Similarly, we can solve by Fourier series the following linear problem which is fundamental to our analysis:

$$\begin{cases} \Delta^2 \boldsymbol{v} + \nabla p = \boldsymbol{f}, & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{v} = 0, & \text{in } \Omega. \end{cases} \tag{7}$$

The linear operator $A$ (which we call the dissipation operator) is characterized by $A\boldsymbol{v} = \boldsymbol{f}$. It can be shown that the self adjoint operator $A \in \mathcal{L}(D(A); H) \cap \mathcal{L}(V; V')$ is closed with $D(A)$ dense in $V \subset H$. Moreover, we can define positive as well as negative powers $A^\alpha, \alpha \in \boldsymbol{R}$ with domain $D(A^\alpha)$. Let us denote by $X_\alpha$ the space,

$$X_\alpha = D(A^{\alpha/4}) = \{\boldsymbol{u} \in \dot{H}^\alpha(\Omega),\ \text{div}\boldsymbol{u} = 0\}.$$

In fact the norm induced by $\dot{H}^\alpha(\Omega)$ is equivalent to the norm $|A^{\alpha/4}\boldsymbol{u}|$ in $X_\alpha$. This means

$$\beta_1 \|\boldsymbol{u}\|_{4\alpha} \le |A^\alpha \boldsymbol{u}| \le \beta_2 \|\boldsymbol{u}\|_{4\alpha}, \quad \forall\, \boldsymbol{u} \in D(A^\alpha), \quad \forall\, \alpha \in \boldsymbol{R}. \tag{8}$$

We can deduce from a theorem due to Lions [7] that $D(A^{1/2}) = V$. Thus,

$$D(A_1) = V = D(A^{1/2}). \tag{9}$$

By Rellich's Lemma [1] $A^{-1}$ as a mapping in $H$ is compact. Hence the spectrum of operator $A$ is discrete with finite multiplicities and can be written explicitly using Fourier series as

$$\lambda_k = \frac{16\pi^4}{L^4}|k|^4, \qquad k = (k_1, k_2).$$

The self adjoint operator $A$ possesses an orthonormal set of eigenfunctions $\{\boldsymbol{w}_j\}_{j=1}^{\infty}$ complete in $H$,

$$A\boldsymbol{w}_j = \lambda_j \boldsymbol{w}_j\,, \qquad \boldsymbol{w}_j \in D(A),\ \forall\, j.$$

The above results can be applied to the Stokes operator $A_1$ as well. The eigenvalues of $A_1$ are

$$\mu_k = \frac{4\pi^2}{L^2}|k|^2, \qquad k = (k_1, k_2).$$

The corresponding orthonormal set of eigenfunctions $\{\hat{\boldsymbol{w}}_j\}_{j=1}^{\infty}$ is complete in $H$ and

$$A_1\hat{\boldsymbol{w}}_j = \lambda_j \hat{\boldsymbol{w}}_j\,, \qquad \hat{\boldsymbol{w}}_j \in D(A_1),\ \forall\, j.$$

**Remark:** For the space periodic case considered in this paper, the Stokes operator $A_1$ is in fact equal to $A^{1/2}$. However since for other types of boundary conditions they are different we prefer to give them distinct notations.

Let us now define the trilinear form $b(\cdot,\cdot,\cdot)$ associated with the inertia terms:

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \sum_{i,j=1}^{2} \int_\Omega \boldsymbol{u}_i D_i \boldsymbol{v}_j \boldsymbol{w}_j\, d\boldsymbol{x}, \qquad D_i = \frac{\partial}{\partial x_i}.$$

It can be easily shown that

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}), \qquad \forall\, \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in V$$

and

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0, \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \in V.$$

In the case of periodic boundary conditions, we have in addition [15, Lemma 3.1]

$$b(\boldsymbol{u},\boldsymbol{u},A_1\boldsymbol{u}) = 0, \qquad \forall\, \boldsymbol{u} \in D(A_1). \tag{10}$$

Using the discrete Hölder inequality and Sobolev embedding theorem we can show that $b(\cdot,\cdot,\cdot)$ is trilinear continuous on $\dot{H}^{m_1}(\Omega) \times \dot{H}^{m_2+1}(\Omega) \times \dot{H}^{m_3}(\Omega)$, $m_i \geq 0$ [15]:

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leq c_0 \|\boldsymbol{u}\|_{m_1} \|\boldsymbol{v}\|_{m_2+1} \|\boldsymbol{w}\|_{m_3},$$

$$m_1 + m_2 + m_3 > 1.$$

The following well known estimate will be used later,

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq c_1 |\boldsymbol{u}|^{1/2} |\nabla \boldsymbol{u}|^{1/2} |\boldsymbol{v}|^{1/2} |\nabla \boldsymbol{v}|^{1/2} |\nabla \boldsymbol{w}|, \qquad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V(\text{ or } \hat{V}), \tag{11}$$

where $c_1$ is a positive constant.

The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator $B$ from $\dot{H}^{m_1}(\Omega) \times \dot{H}^{m_2+1}(\Omega)$ into $(\dot{H}^{m_3}(\Omega))'$. In particular, for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$, $B(\boldsymbol{u}, \boldsymbol{v}) \in V'$ will be defined by

$$< B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w} >_{V' \times V} = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \qquad \forall\, \boldsymbol{w} \in V.$$

Similarly, we define $\hat{B}(\boldsymbol{u}, \boldsymbol{v}) \in \hat{V}'$ by

$$< \hat{B}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w} >_{\hat{V}' \times \hat{V}} = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \qquad \forall\, \boldsymbol{w} \in \hat{V}.$$

Using the operators defined above we can write the regularized system (1)-(5) in the evolution form:

$$\begin{cases} \dfrac{d\boldsymbol{u}_\epsilon}{dt} + \epsilon A \boldsymbol{u}_\epsilon + \nu A_1 \boldsymbol{u}_\epsilon + B(\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon) = \boldsymbol{f}, \quad t > 0, \\ \boldsymbol{u}_\epsilon(0) = \boldsymbol{u}_{\epsilon 0}. \end{cases} \tag{12}$$

The existence and uniqueness theorems for initial value problem (12) can be found in [10, 9]. The main result in this work is:

**Theorem 2.1**

(i) *Let* $\boldsymbol{f} \in L^2(0,T;V')$ *and* $u_0 \in H$ *be given. Then there exists a unique weak solution of (12) which satisfies* $\boldsymbol{u}_\epsilon \in C([0,T];H) \cap L^2(0,T;V)$, $\forall\, T > 0$.

(ii) *Let* $\boldsymbol{f} \in L^2(0,T;H)$ *and* $u_0 \in V$ *be given. Then there exists a unique strong solution of (12) which satisfies* $\boldsymbol{u}_\epsilon \in C([0,T];V) \cap L^2(0,T;D(A))$, $\forall\, T > 0$.

□

Let us denote by $W_\epsilon(t)$ the nonlinear semigroup associated with the solution of the regularized system (12). This means $W_\epsilon(t, 0; \boldsymbol{u}_{\epsilon 0}) = \boldsymbol{u}_\epsilon(t)$ for all $t \geq 0$.

Notice that the conventional Navier-Stokes system can be written in the evolution form:

$$\begin{cases} \dfrac{d\boldsymbol{u}}{dt} + \nu A_1 \boldsymbol{u} + \hat{B}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}, \quad t > 0, \\ \\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases} \tag{13}$$

The following unique solvability theorem for the system (13) is well known.

**Theorem 2.2**

(i) *Let* $\boldsymbol{f} \in L^2(0,T;\hat{V}')$ *and* $u_0 \in H$ *be given. Then there exists a unique weak solution of (13) which satisfies* $\boldsymbol{u} \in C([0,T]; H) \cap L^2(0,T;\hat{V})$, $\forall\, T > 0$ *(Lions and Prodi [8]).*

(ii) *Let* $\boldsymbol{f} \in L^2(0,T;H)$ *and* $u_0 \in \hat{V}$ *be given. Then there exists a unique strong solution of (13) which satisfies* $\boldsymbol{u} \in C([0,T]; \hat{V}) \cap L^2(0,T; D(A_1))$, $\forall\, T > 0$ *(Kiselev and Ladyzhenskaya [5]).*

□

We will denote by $W(t) : \boldsymbol{u}_0 \to \boldsymbol{u}(t)$ the nonlinear semigroup associated with the solution of Navier-Stokes equations.

Let us introduce the following result [10, 9] which is needed for the convergence of $\boldsymbol{u}_\epsilon$ as the regularized parameter $\epsilon \to 0$.

**Proposition 2.1** *Let* $\{\boldsymbol{u}_k\}$ *be a sequence of vectorfields that converge weakly in* $L^2(0,T;\hat{V})$, *weak-Star in* $L^\infty(0,T;H)$ *and strongly in* $L^2(0,T;H)$. *Then, the following limits are obtained for any vector function* $\boldsymbol{w}$ *in* $\mathcal{Y} = \{\boldsymbol{w} \in C(0,T;\hat{V}); \boldsymbol{w}' \in L^2(0,T;H)\}$

$$(a) \quad \lim_{k\to\infty} \int_0^T (\boldsymbol{u}_k(t), \boldsymbol{w}'(t))dt = \int_0^T (\boldsymbol{u}(t), \boldsymbol{w}'(t))dt,$$

$$(b) \quad \lim_{k\to\infty} \int_0^T (\nabla \boldsymbol{u}_k(t), \nabla \boldsymbol{w}(t))dt = \int_0^T (\nabla \boldsymbol{u}(t), \nabla \boldsymbol{w}(t))dt,$$

$$(c) \quad \lim_{k\to\infty} \int_0^T b(\boldsymbol{u}_k(t), \boldsymbol{u}_k(t), \boldsymbol{w}(t))dt = \int_0^T b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{w}(t))dt.$$

□

# 3 Uniform Estimates for the Solutions

In this section we will establish various estimates uniform in $\epsilon$ for the solution of regularized Navier-Stokes equations. These bounds will be used to establish the limit of these solutions to the conventional Navier-Stokes equations.

**Lemma 3.1** *Let $f \in L^2(0,T;H)$, then the weak solution $\boldsymbol{u}_\epsilon(t)$ of the regularized Navier-Stokes equations satisfy*

$$\text{(i)} \sup_{t\in[0,T]} |\boldsymbol{u}_\epsilon(t)|^2 \le d_1,$$

$$\text{(ii)} \int_0^T |\nabla \boldsymbol{u}_\epsilon(t)|^2 dt \le d_2,$$

*where $d_1, d_2$ are both independent of $\epsilon$.*

**Proof:** By taking the inner product of (12) with $\boldsymbol{u}_\epsilon(t)$, we obtain the energy equality:

$$\frac{d}{dt}|\boldsymbol{u}_\epsilon|^2 + 2\epsilon\|\boldsymbol{u}_\epsilon\|^2 + 2\nu|\nabla \boldsymbol{u}_\epsilon|^2 = 2(\boldsymbol{f}, \boldsymbol{u}_\epsilon).$$

Here we have used the fact that $b(\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon) = 0$. By applying Young's inequality and the Poincaré Lemma $|\nabla \boldsymbol{u}_\epsilon|^2 \ge \mu_1 |\boldsymbol{u}_\epsilon|^2$, we get

$$\frac{d}{dt}|\boldsymbol{u}_\epsilon|^2 + 2\epsilon\|\boldsymbol{u}_\epsilon\|^2 + \nu|\nabla \boldsymbol{u}_\epsilon|^2 \le \frac{|\boldsymbol{f}|^2}{\nu\mu_1}, \tag{14}$$

where $\mu_1 = 4\pi^2/L^2$ is the smallest eigenvalue of the Stokes operator $A_1$. If we drop the positive term associated with $\epsilon$, we obtain

$$\frac{d}{dt}|\boldsymbol{u}_\epsilon|^2 + \nu\mu_1|\boldsymbol{u}_\epsilon|^2 \le \frac{|\boldsymbol{f}|^2}{\nu\mu_1}.$$

Hence, by integrating the above inequality from 0 to $t$, we get

$$|\boldsymbol{u}_\epsilon(t)|^2 \le |\boldsymbol{u}_{\epsilon 0}|^2 e^{-\alpha t} + \frac{1}{\alpha}\int_0^t |\boldsymbol{f}(s)|^2 e^{-\alpha(t-s)}\, ds,$$

with $\alpha = \nu\mu_1$. That is,

$$\sup_{t\in[0,T]} |\boldsymbol{u}_\epsilon(t)|^2 \le d_1.$$

$$\text{with} \quad d_1 = d_1(\boldsymbol{u}_{\epsilon 0}, \boldsymbol{f}, \alpha, T) = |\boldsymbol{u}_{\epsilon 0}|^2 + \frac{1}{\alpha}\int_0^T |\boldsymbol{f}(s)|^2 ds.$$

If we drop the term $2\epsilon\|\boldsymbol{u}_\epsilon\|^2$ in (14) and integrate from 0 to $T$,

$$|\boldsymbol{u}_\epsilon(T)|^2 + \nu\int_0^T |\nabla \boldsymbol{u}_\epsilon|^2 dt \leq |\boldsymbol{u}_{\epsilon 0}|^2 + \frac{1}{\alpha}\int_0^T |\boldsymbol{f}|^2 dt. \tag{15}$$

This implies

$$\int_0^T |\nabla \boldsymbol{u}_\epsilon|^2 dt \leq \frac{1}{\nu}[\ |\boldsymbol{u}_{\epsilon 0}|^2 + \frac{1}{\alpha}\int_0^T |\boldsymbol{f}|^2 dt\ ].$$

That is

$$\int_0^T |\nabla \boldsymbol{u}_\epsilon(t)|^2 dt \leq d_2, \qquad d_2 = \frac{d_1}{\nu}.$$

Notice that $d_1$ and $d_2$ do not depend on $\epsilon$.

□

**Corollary 3.1** *Let $\boldsymbol{f} \in H$ be independent of time then,*

$$\text{(i)}|\boldsymbol{u}_\epsilon(t)|^2 \leq |\boldsymbol{u}_\epsilon(0)|^2 e^{-\alpha t} + \rho_0^2(1 - e^{-\alpha t}), \quad \rho_0 = \frac{|\boldsymbol{f}|}{\alpha}, \ \forall\, t > 0,$$
$$\text{(ii)}\int_0^T |\nabla \boldsymbol{u}_\epsilon|^2 dt \leq \frac{1}{\nu}(\ |\boldsymbol{u}_{\epsilon 0}|^2 + \frac{|\boldsymbol{f}|^2 T}{\alpha}),\ T > 0.$$

**Proof:** By integrating (14) from 0 to $t$ with $\boldsymbol{f} \in H$, we obtain result (i) with $\rho_0 = |\boldsymbol{f}|/\alpha$. Result (ii) is a direct consequence of (15).

□

**Lemma 3.2** *Let $\boldsymbol{f} \in L^2(0, T; H)$, then the strong solution $\boldsymbol{u}_\epsilon(t)$ of the regularized Navier-Stokes equations satisfy*

$$\text{(i)} \sup_{t\in[0,T]} |\nabla \boldsymbol{u}_\epsilon(t)|^2 \leq d_3,$$
$$\text{(ii)} \int_0^T |A_1 \boldsymbol{u}_\epsilon(t)|^2 dt \leq d_4,$$

*where $d_3, d_4$ are both independent of $\epsilon$.*

**Proof:** We consider the strong solutions and take the inner product of (12) with $A_1 \boldsymbol{u}_\epsilon$ obtaining

$$\frac{d}{dt}|\nabla \boldsymbol{u}_\epsilon|^2 + 2\epsilon|A^{3/4}\boldsymbol{u}_\epsilon|^2 + 2\nu|A_1\boldsymbol{u}_\epsilon|^2 = 2(\boldsymbol{f}, A_1\boldsymbol{u}_\epsilon).$$

Here we have used the fact that $b(\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon, A_1\boldsymbol{u}_\epsilon) = 0$ and

$$\begin{aligned}
(A\boldsymbol{u}_\epsilon, A_1\boldsymbol{u}_\epsilon) &= (A^{1/4}A^{3/4}\boldsymbol{u}_\epsilon, A^{1/2}\boldsymbol{u}_\epsilon) \\
&= (A^{3/4}\boldsymbol{u}_\epsilon, A^{3/4}\boldsymbol{u}_\epsilon) \\
&= |A^{3/4}\boldsymbol{u}_\epsilon|^2
\end{aligned}$$

since $D(A_1) = D(A^{1/2})$ and both $A, A_1$ are self adjoint. By applying Young's inequality, we get

$$\frac{d}{dt}|\nabla \boldsymbol{u}_\epsilon|^2 + 2\epsilon\,|A^{3/4}\boldsymbol{u}_\epsilon|^2 + \nu|A_1\boldsymbol{u}_\epsilon|^2 \le \frac{|\boldsymbol{f}|^2}{\nu}. \tag{16}$$

We can drop the positive term $2\epsilon\,|A^{3/4}\boldsymbol{u}_\epsilon|^2$ and use the inequality $|A_1\boldsymbol{u}_\epsilon|^2 \ge \mu_1|\nabla\boldsymbol{u}_\epsilon|^2$ to get

$$\frac{d}{dt}|\nabla \boldsymbol{u}_\epsilon|^2 + \nu\mu_1|\nabla \boldsymbol{u}_\epsilon|^2 \le \frac{|\boldsymbol{f}|^2}{\nu}. \tag{17}$$

Integrating the above inequality from 0 to $t$, we get

$$|\nabla \boldsymbol{u}_\epsilon(t)|^2 \le |\nabla \boldsymbol{u}_{\epsilon 0}|^2 e^{-\alpha t} + \frac{1}{\nu}\int_0^t |\boldsymbol{f}(s)|^2\, e^{-\alpha(t-s)}\, ds,$$

where $\alpha = \nu\mu_1$. This gives

$$\sup_{t\in[0,T]} |\nabla \boldsymbol{u}_\epsilon(t)|^2 \le d_3.$$

$$\text{with} \quad d_3 = d_3(\boldsymbol{u}_{\epsilon 0}, \boldsymbol{f}, \nu, T) = |\nabla \boldsymbol{u}_{\epsilon 0}|^2 + \frac{1}{\nu}\int_0^T |\boldsymbol{f}(s)|^2 ds.$$

We can drop the term $2\epsilon\,|A^{3/4}\boldsymbol{u}_\epsilon|^2$ in (16) and integrate from 0 to $T$ to get

$$|\nabla \boldsymbol{u}_\epsilon(T)|^2 + \nu\int_0^T |A_1\boldsymbol{u}_\epsilon|^2 dt \le |\nabla \boldsymbol{u}_{\epsilon 0}|^2 + \frac{1}{\nu}\int_0^T |\boldsymbol{f}|^2 dt. \tag{18}$$

This implies

$$\int_0^T |A_1\boldsymbol{u}_\epsilon|^2 dt \le \frac{1}{\nu}[\ |\nabla \boldsymbol{u}_{\epsilon 0}|^2 + \frac{1}{\nu}\int_0^T |\boldsymbol{f}|^2 dt\ ].$$

That is

$$\int_0^T |A_1\boldsymbol{u}_\epsilon(t)|^2 dt \le d_4, \qquad d_4 = \frac{d_3}{\nu}.$$

Notice that $d_3$ and $d_4$ do not depend on $\epsilon$.

$\square$

From this proof we can easily deduce the following results.

**Corollary 3.2** *Let $f \in H$ be independent of time then the strong solution $u_\epsilon(t)$ satisfies:*

$$(\mathrm{i})|\nabla u_\epsilon(t)|^2 \le |\nabla u_\epsilon(0)|^2 e^{-\alpha t} + \rho_1^2(1-e^{-\alpha t}), \quad \rho_1^2 = \frac{|f|^2}{\nu\alpha}, \ \forall\, t>0,$$
$$(\mathrm{ii}) \int_0^T |A_1 u_\epsilon|^2 dt \le \frac{1}{\nu}(\,|\nabla u_{\epsilon 0}|^2 + \frac{|f|^2 T}{\nu}),\ T>0.$$

$\square$

# 4 Limit to the Navier-Stokes Equations

In this section we will establish the convergence of the strong solution of the regularized Navier-Stokes equations as $\epsilon \to 0$. In [10, 9], a similar convergence result for the weak solution was established under certain conditions. In this section we will prove the convergence *without any assumption* on the bound for the solutions.

**Theorem 4.1** *Let $u_\epsilon(t)$ be the strong solution given by Theorem 2.1. Then as $\epsilon \to 0$, the solution $u_\epsilon$ converges to a unique limit which is the strong solution of the Navier-Stokes equations.*

**Proof:** We need three forms of convergence for appropriate subsequences. Namely,

$$\begin{array}{llll} (\mathrm{i})\ u_{\epsilon_{k'}} \longrightarrow u & \text{in} & L^2(0,T;\hat{V}) & \text{weakly;} \\ (\mathrm{ii})\ u_{\epsilon_{k'}} \longrightarrow u & \text{in} & L^\infty(0,T;H) & \text{weak-star;} \\ (\mathrm{iii})\ u_{\epsilon_{k'}} \longrightarrow u & \text{in} & L^2(0,T;H) & \text{strongly,} \end{array} \tag{19}$$

as $\epsilon_{k'} \to 0$.

It follows from Lemma 3.2 that $u_{\epsilon_k} \in L^2(0,T;D(A_1)) \cap L^\infty(0,T;\hat{V})$ with bounds independent of $\epsilon_k$. This easily implies (i) and (ii). (In fact, better convergence results are obtained.) The strong convergence result (iii) can be established as follows:

Let $\{\boldsymbol{u}_{\epsilon_k}\}$ and $\{\boldsymbol{u}_{\epsilon_l}\}$ be two sequences of strong solutions of (12) corresponding to the same initial data with $\epsilon_k = \frac{1}{k}$ and $\epsilon_l = \frac{1}{l}$, respectively. (i.e. $\epsilon_k, \epsilon_l \to 0$ as $k, l \to \infty$.) We denote by $\boldsymbol{u}_\epsilon(t) = \boldsymbol{u}_{\epsilon_k}(t) - \boldsymbol{u}_{\epsilon_l}(t)$ so that $\boldsymbol{u}_\epsilon(0) = 0$. By subtracting the equation (12) for $\boldsymbol{u}_{\epsilon_l}$ from that for $\boldsymbol{u}_{\epsilon_k}$, we get

$$\boldsymbol{u}_\epsilon' + \epsilon_k A\boldsymbol{u}_{\epsilon_k} - \epsilon_l A\boldsymbol{u}_{\epsilon_l} + \nu A_1 \boldsymbol{u}_\epsilon + B(\boldsymbol{u}_{\epsilon_k}, \boldsymbol{u}_{\epsilon_k}) - B(\boldsymbol{u}_{\epsilon_l}, \boldsymbol{u}_{\epsilon_l}) = 0 \tag{20}$$

Taking the inner product of (20) with $\boldsymbol{u}_\epsilon$, we get

$$(\boldsymbol{u}_\epsilon', \boldsymbol{u}_\epsilon) + \epsilon_k (A\boldsymbol{u}_{\epsilon_k}, \boldsymbol{u}_\epsilon) - \epsilon_l (A\boldsymbol{u}_{\epsilon_l}, \boldsymbol{u}_\epsilon) + \nu(A_1\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon) + b(\boldsymbol{u}_{\epsilon_k}, \boldsymbol{u}_{\epsilon_k}, \boldsymbol{u}_\epsilon) - b(\boldsymbol{u}_{\epsilon_l}, \boldsymbol{u}_{\epsilon_l}, \boldsymbol{u}_\epsilon) = 0.$$

By using the fact that $(A\boldsymbol{u}, \boldsymbol{v}) = (A_1\boldsymbol{u}, A_1\boldsymbol{v})$ and the properties of the trilinear form $b$ established earlier, we obtain

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_\epsilon|^2 + \epsilon_k (A_1\boldsymbol{u}_{\epsilon_k}, A_1\boldsymbol{u}_\epsilon) - \epsilon_l (A_1\boldsymbol{u}_{\epsilon_l}, A_1\boldsymbol{u}_\epsilon) + \nu|\nabla\boldsymbol{u}_\epsilon|^2 - b(\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon, \boldsymbol{u}_{\epsilon_k}) = 0.$$

Thus

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_\epsilon|^2 + \nu|\nabla\boldsymbol{u}_\epsilon|^2 \le \epsilon_k |(A_1\boldsymbol{u}_{\epsilon_k}, A_1\boldsymbol{u}_\epsilon)| + \epsilon_l |(A_1\boldsymbol{u}_{\epsilon_l}, A_1\boldsymbol{u}_\epsilon)| + |b(\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon, \boldsymbol{u}_{\epsilon_k})|. \tag{21}$$

The trilinear term can be estimated using (11) as,

$$\begin{aligned} |b(\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon, \boldsymbol{u}_{\epsilon_k})| &\le c_1 |\boldsymbol{u}_\epsilon|\, |\nabla\boldsymbol{u}_\epsilon|\, |\nabla\boldsymbol{u}_{\epsilon_k}| \\ &\le \frac{\nu}{2}|\nabla\boldsymbol{u}_\epsilon|^2 + \frac{c_1^2}{2\nu}|\boldsymbol{u}_\epsilon|^2\, |\nabla\boldsymbol{u}_{\epsilon_k}|^2. \end{aligned}$$

Substituting the above result into (21), we obtain

$$\frac{d}{dt}|\boldsymbol{u}_\epsilon|^2 + \nu|\nabla\boldsymbol{u}_\epsilon|^2 \le 2\epsilon_k |A_1\boldsymbol{u}_{\epsilon_k}|\, |A_1\boldsymbol{u}_\epsilon| + 2\epsilon_l |A_1\boldsymbol{u}_{\epsilon_l}|\, |A_1\boldsymbol{u}_\epsilon| + \frac{c_1^2}{\nu}|\boldsymbol{u}_\epsilon|^2\, |\nabla\boldsymbol{u}_{\epsilon_k}|^2.$$

We drop the positive term $\nu|\nabla\boldsymbol{u}_\epsilon|^2$ and integrate this equation from 0 to $t$, and noting that $\boldsymbol{u}_\epsilon(0) = 0$, we get

$$\begin{aligned} |\boldsymbol{u}_\epsilon(t)|^2 &\le 2\epsilon_k \int_0^t |A_1\boldsymbol{u}_{\epsilon_k}|\, |A_1\boldsymbol{u}_\epsilon| d\tau + 2\epsilon_l \int_0^t |A_1\boldsymbol{u}_{\epsilon_l}|\, |A_1\boldsymbol{u}_\epsilon| d\tau \\ &\quad + \frac{c_1^2}{\nu}\int_0^t |\boldsymbol{u}_\epsilon|^2\, |\nabla\boldsymbol{u}_{\epsilon_k}|^2 d\tau. \end{aligned} \tag{22}$$

From Lemma 3.2, we have $\boldsymbol{u}_{\epsilon_k} \in L^2(0,T;D(A_1))$ with bound independent of $\epsilon_k$. Hence by applying the Schwartz inequality we have

$$\begin{aligned}\epsilon_k \int_0^t |A_1 \boldsymbol{u}_{\epsilon_k}|\,|A_1 \boldsymbol{u}_\epsilon| d\tau &\le \epsilon_k \Big(\int_0^t |A_1 \boldsymbol{u}_{\epsilon_k}|^2 d\tau\Big)^{1/2} \Big(\int_0^t |A_1 \boldsymbol{u}_\epsilon|^2 d\tau\Big)^{1/2} \\ &\le \epsilon_k (3^{1/2} d_4).\end{aligned}$$

Similarly we have

$$\epsilon_l \int_0^t |A_1 \boldsymbol{u}_{\epsilon_l}|\,|A_1 \boldsymbol{u}_\epsilon| d\tau \le \epsilon_l (3^{1/2} d_4).$$

Since from Lemma 3.2 $\boldsymbol{u}_{\epsilon_k} \in L^\infty(0,T;\hat{V})$ with bound independent of $\epsilon_k$,

$$\int_0^t |\boldsymbol{u}_\epsilon|^2\,|\nabla \boldsymbol{u}_{\epsilon_k}|^2 d\tau \le d_3 \int_0^t |\boldsymbol{u}_\epsilon|^2 d\tau.$$

Combining all of the above results into inequality (22) we get,

$$|\boldsymbol{u}_\epsilon(t)|^2 \le 2 \cdot 3^{1/2} d_4 (\epsilon_k + \epsilon_l) + \frac{c_1^2 d_3}{\nu} \int_0^t |\boldsymbol{u}_\epsilon(\tau)|^2 d\tau. \tag{23}$$

Notice that here $c_1$, $d_3$ and $d_4$ are all independent of $\epsilon$. Now, if we denote $y(t) = \int_0^t |\boldsymbol{u}_\epsilon(\tau)|^2 d\tau$ then (23) becomes

$$\begin{cases} \dfrac{dy}{dt} \le 2 \cdot 3^{1/2} d_4 (\epsilon_k + \epsilon_l) + \dfrac{c_1^2 d_3}{\nu} y(t) \\ y(0) = 0. \end{cases}$$

From this we can deduce that,

$$y(t) \le 3^{1/2} \nu (\epsilon_k + \epsilon_l) \frac{d_3}{d_4} \left[ \exp\left( \frac{c_1^2 d_3}{\nu} t \right) - 1 \right], \qquad t \in [0,T].$$

Hence,

$$\int_0^T |\boldsymbol{u}_\epsilon(\tau)|^2 d\tau \longrightarrow 0, \qquad \text{as } \epsilon_k, \epsilon_l \to 0.$$

That is

$$\|\boldsymbol{u}_{\epsilon_k} - \boldsymbol{u}_{\epsilon_l}\|_{L^2(0,T;H)} \longrightarrow 0, \qquad \text{as } \epsilon_k, \epsilon_l \to 0.$$

This means $\{\boldsymbol{u}_{\epsilon_k}\}$ is a Cauchy sequence in $L^2(0,T;H)$ and there exists a limit $\boldsymbol{u}$ in $L^2(0,T;H)$ such that

$$\|\boldsymbol{u}_{\epsilon_k} - \boldsymbol{u}\|_{L^2(0,T;H)} \longrightarrow 0 \quad \text{as } \epsilon_k \to 0.$$

This completes the proof of the strong convergence result (iii) of (19).

Let us now establish the limit of the equation (12) as $\epsilon_k \to 0$. Let us take a test element $\phi$ such that

$$\phi \in C(0,T;V) \quad \text{and} \quad \phi' \in L^2(0,T;H).$$

Taking the inner product with (12), we get

$$(u'_{\epsilon_k}, \phi) + \epsilon_k (A u_{\epsilon_k}, \phi) + \nu (A_1 u_{\epsilon_k}, \phi) + b(u_{\epsilon_k}, u_{\epsilon_k}, \phi) = (f, \phi),$$

Integrating with respect to $t$ and then integrating by parts we get,

$$\begin{aligned}(u_{\epsilon_k}(T), \phi(T)) - (u_{\epsilon_k}(0), \phi(0)) - \int_0^T (u_{\epsilon_k}, \phi')dt \\ +\epsilon_k \int_0^T (A_1 u_{\epsilon_k}, A_1 \phi)dt + \nu \int_0^T (\nabla u_{\epsilon_k}, \nabla \phi)dt \\ + \int_0^T b(u_{\epsilon_k}, u_{\epsilon_k}, \phi)dt = \int_0^T (f, \phi)dt. \end{aligned} \tag{24}$$

Let us choose $\psi = \phi \in \mathcal{D}(0,T;V)$. Using the convergence Proposition 2.1 we can take the equation (24) to the limit as $\epsilon_k \to 0$,

$$-\int_0^T (u, \psi')dt + \nu \int_0^T (\nabla u, \nabla \psi)dt + \int_0^T b(u, u, \psi)dt = \int_0^T (f, \psi)dt.$$

Here the term

$$\epsilon_k \int_0^T (A_1 u_{\epsilon_k}, A_1 \psi)dt \longrightarrow 0,$$

as $\epsilon_k \to 0$ since

$$\int_0^T (A_1 u_{\epsilon_k}, A_1 \psi)dt \leq \text{const. independent of } \epsilon_k.$$

Since $u \in L^2(0,T;D(A_1)) \cap L^\infty(0,T;\hat{V})$, we can conclude that $u$ is indeed the strong solution for the conventional Navier-Stokes equations.

□

From Lemma 3.1, we can deduce that the weak solutions $u_{\epsilon_k} \in L^2(0,T;\hat{V}) \cap L^\infty(0,T;H)$ are bounded independent of $\epsilon_k$. Hence the convergence results (i) and (ii) are satisfied by

the weak solutions. However, such convergences are not sufficient to conclude the strong convergence in $L^2(0,T;H)$. Therefore, Theorem 4.1 is not valid for weak solutions. We can nevertheless establish a convergence theorem for the weak solutions provided they satisfy a certain bound. The following theorem is proved in [10, 9]:

**Theorem 4.2** *Let $\boldsymbol{u}_\epsilon$ be the weak solution of problem (12) given by Theorem 2.1 with $\boldsymbol{f} \in L^2(0,T;H)$. Moreover, $\boldsymbol{u}_\epsilon \in L^\infty(0,T;L^{3+\delta}(\Omega))$ uniformly in $\epsilon$ ($\delta > 0$) and $\nu > c$, where $c$ is a positive constant. Then, for $\epsilon \to 0$, $\boldsymbol{u}_\epsilon$ approaches a unique limit $\boldsymbol{u}$ which is the weak solution of the Navier-Stokes equations.*

□

# 5 Convergence of T-time Periodic Solution

In this section we will show that the $T$-time periodic solution of the regularized Navier-Stokes equations converges in the limit to a $T$-time periodic solution of Navier-Stokes equations. Existence theorems for $T$-time periodic solutions to the regularized Navier-Stokes equations were established in [10, 9]. For similar existence theorems for conventional Navier-Stokes equations see [12, 17, 14]. Let us first give a definition for the time periodic solution.

**Definition 5.1** *Let $\boldsymbol{f}(t) \in L^2_{loc}(-\infty,+\infty;H)$ and be periodic with period $T$. Then $\boldsymbol{u}_\epsilon(t)$ is said to be a $T$-time periodic solution of regularized Navier-Stokes equations if it satisfies $\boldsymbol{u}_\epsilon(t) \in L^\infty(-\infty,+\infty;H) \cap L^2_{loc}(-\infty,+\infty;V)$, $T$-periodic in time and*

$$\int_{-\infty}^{+\infty}\{-(\boldsymbol{u}_\epsilon(t),\boldsymbol{\psi}'(t)) + \epsilon(\Delta\boldsymbol{u}_\epsilon(t),\Delta\boldsymbol{\psi}(t)) + \nu(\nabla\boldsymbol{u}_\epsilon(t),\nabla\boldsymbol{\psi}(t))$$

$$+ b(\boldsymbol{u}_\epsilon(t),\boldsymbol{u}_\epsilon(t),\boldsymbol{\psi}(t))\}dt = \int_{-\infty}^{+\infty}(\boldsymbol{f}(t),\boldsymbol{\psi}(t))dt, \quad \forall\, \boldsymbol{\psi} \in \mathcal{Y} \tag{25}$$

$$\textit{with} \quad \mathcal{Y} = \{\boldsymbol{\psi} \in C_0(-\infty,+\infty;V) \textit{ and } \boldsymbol{\psi}' \in L^2(-\infty,+\infty;H)\}.$$

Here $C_0(-\infty,+\infty;V)$ denotes the set of continuous $V$-valued functions that vanish outside a compact time interval.

In [10, 9], it has been proven that a necessary and sufficient condition for a $T$-periodic solution to be a weak solution in $(-\infty,+\infty)$ is that it must be a weak solution in the interval $[0,T]$. The regularity of periodic solutions can be obtained by noting that $\exists t_1 \in (-\infty,+\infty)$ such that $u_\epsilon(x,t_1) \in V$. We use this as the initial value and study the regularity in $[t_1,\infty)$ by standard methods [15]. Repeating this in each interval we deduce the regularity of the periodic solution in the entire interval $(-\infty,+\infty)$ as $u_\epsilon(t) \in L^\infty(-\infty,+\infty;V) \cap L^2_{loc}(-\infty,+\infty;D(A))$. Let us now state the main theorem:

**Theorem 5.1** *Let $f(t) \in L^2_{loc}(-\infty,+\infty;H)$ be given and $u_\epsilon(t) \in L^\infty(-\infty,+\infty;V) \cap L^2_{loc}(-\infty,+\infty;D(A))$ be the $T$-time periodic solution of the regularized Navier-Stokes equations. Then as $\epsilon \to 0$, $u_\epsilon(t)$ converges to a limit $u \in L^\infty(-\infty,+\infty;\hat{V}) \cap L^2_{loc}(-\infty,+\infty;D(A_1))$ which is a $T$-time periodic solution of the Navier-Stokes equations*

**Proof:** Since $u_\epsilon(t)$ is a $T$-periodic solution which satisfies (25), for any $k \in Z$ we can choose a test function $\psi(t) = \psi_0(t)$ so that it vanishes outside the compact interval $(kT,(k+1)T)$. Thus,

$$\int_{kT}^{(k+1)T} \{-(u_\epsilon(t),\psi_0'(t)) + \epsilon(\Delta u_\epsilon(t),\Delta\psi_0(t)) + \nu(\nabla u_\epsilon(t),\nabla\psi_0(t))$$
$$+b(u_\epsilon(t),u_\epsilon(t),\psi_0(t))\}dt = \int_{kT}^{(k+1)T}(f(t),\psi_0(t))dt, \quad \forall\, k \in Z,$$

with $u_\epsilon(t) \in L^\infty(kT,(k+1)T;V) \cap L^2(kT,(k+1)T;D(A))$. In particular if we let $k=0$, we obtain the strong solution in the sub-interval $(0,T)$.

Let $\{u_{\epsilon_k}\}$ be a sequence of such solutions with $\epsilon_k = 1/k$. Then,

$$\int_0^T \{-(u_{\epsilon_k}(t),\psi_0'(t)) + \epsilon_k(\Delta u_{\epsilon_k}(t),\Delta\psi_0(t)) + \nu(\nabla u_{\epsilon_k}(t),\nabla\psi_0(t))$$
$$+b(u_{\epsilon_k}(t),u_{\epsilon_k}(t),\psi_0(t))\}dt = \int_0^T (f(t),\psi_0(t))dt,$$
$$\forall \quad \psi_0 \in C(0,T;V) \quad \psi_0' \in L^2(0,T;H).$$

From Lemma 3.2, we have the strong solution $u_{\epsilon_k} \in L^2(0,T;D(A_1)) \cap L^\infty(0,T;\hat{V})$ with bounds uniform in $\epsilon_k$. Thus, by applying the convergence Proposition 2.1 we obtain

$$\int_0^T \{-(u(t),\psi_0'(t)) + \nu(\nabla u(t),\nabla\psi_0(t))$$

$$+b(u(t),u(t),\psi_0(t))\}dt = \int_0^T (f(t),\psi_0(t))dt,$$

$$\forall \quad \psi_0 \in C(0,T;V)\ \psi_0' \in L^2(0,T;H).$$

Here

$$\epsilon_k \int_0^T (\Delta u_{\epsilon_k}, \Delta\psi_0)dt \longrightarrow 0$$

as $\epsilon_k \to 0$.

Now, by periodicity of the strong solution we get

$$\int_{kT}^{(k+1)T} \{-(u(t),\psi_k'(t)) + \nu(\nabla u(t),\nabla\psi_k(t))$$

$$+ b(u(t),u(t),\psi_k(t))\}dt = \int_{kT}^{(k+1)T} (f(t),\psi_k(t))dt, \tag{26}$$

for each sub-interval $(kT,(k+1)T)$. Here we denote $\psi_k(t) = \psi_0(t-kT)$.

The above result implies that $u(t) \in L^\infty(kT,(k+1)T;\hat{V}) \cap L^2(kT,(k+1)T;D(A_1))$ is a strong solution in each sub-interval. Hence, by choosing a suitable $\psi(t)$ and summing over $k$, we get (25) with $\epsilon = 0$. This completes the proof.

□

# 6 Upper Semicontinuity of Attractors

In this section we will show that the compact global attractor $\Lambda_\epsilon$ associated with the semigroup $W_\epsilon(t,\cdot)$ converges to the compact global attractor $\Lambda$ associated with the conventional Navier-Stokes system. We prove this result by establishing the upper semicontinuity of $\Lambda_\epsilon$ at $\epsilon = 0$. In this section $f \in H$ and is independent of time.

We begin by showing that $W_\epsilon(t,\cdot)$ admits a compact attractor $\Lambda_\epsilon$ in $H$ for each $\epsilon \geq 0$. According to Ladyzhenskaya [6], the formal definition of a global attractor is as follows: A set $\Lambda_\epsilon \subset H$ is called the **global attractor** for the semigroup $W_\epsilon(t,\cdot)$ if

(i) $\Lambda_\epsilon$ is bounded in $H$;

(ii) $\Lambda_\epsilon$ is an invariant set, $W_\epsilon(t,0;\Lambda_\epsilon) = \Lambda_\epsilon, \quad \forall\, t \in \boldsymbol{R}$;

(iii) $\Lambda_\epsilon$ is a compact set that attracts the bounded sets of $H$.

For a study on the global attractor to the Navier-Stokes equations the reader is refered to Constantin, Foias and Temam [3] and Temam [16].

It follows from Corollary 3.1 that for $\boldsymbol{f} \in H$ we have

$$|\boldsymbol{u}_\epsilon(t)|^2 \leq |\boldsymbol{u}_\epsilon(0)|^2 e^{-\alpha t} + \rho_0^2 [1 - e^{-\alpha t}], \quad t > 0, \tag{27}$$

where $\alpha = \nu\mu_1$ and $\rho_0 = |\boldsymbol{f}|/\alpha$. Note that both $\alpha$ and $\rho_0$ are independent of $\epsilon$. Hence, for any ball $B_{R_0} = \{\boldsymbol{u}_\epsilon(0) \in H;\ |\boldsymbol{u}_\epsilon(0)| \leq R_0\}$, there is a ball $B_{r_0}$ in $H$ centered at origin with radius $r_0 > \rho_0$ ($R_0 > r_0$) such that

$$W_\epsilon(t, 0; B_{R_0}) \subset B_{r_0}, \qquad \text{for } t \geq t_0(B_{R_0}) = \frac{1}{\alpha} \ln \frac{R_0^2 - \rho_0^2}{r_0^2 - \rho_0^2}.$$

The ball $B_{r_0}$ is said to be **exponentially absorbing** and **invariant** under the action of the map $W_\epsilon(t, 0; \cdot)$.

Recalling the inequality (14), we have

$$\frac{d}{dt}|\boldsymbol{u}_\epsilon|^2 + 2\epsilon\|\boldsymbol{u}_\epsilon\|^2 + \nu|\nabla \boldsymbol{u}_\epsilon|^2 \leq \frac{|\boldsymbol{f}|^2}{\alpha}. \tag{28}$$

Integrating from $t$ to $t+1$, we obtain for $\boldsymbol{u}_0 \in B_{R_0}$

$$\int_t^{t+1} |\nabla \boldsymbol{u}_\epsilon|^2 d\tau \leq \frac{1}{\nu}\left(r_0^2 + \frac{|\boldsymbol{f}|^2}{\alpha}\right), \quad \forall\, t \geq t_0(B_{R_0}). \tag{29}$$

Now, we recall the inequality (17),

$$\frac{d}{d\tau}|\nabla \boldsymbol{u}_\epsilon|^2 + \nu\mu_1|\nabla \boldsymbol{u}_\epsilon|^2 \leq \frac{|\boldsymbol{f}|^2}{\nu}.$$

We drop the positive terms $\nu\mu_1|\nabla \boldsymbol{u}_\epsilon|^2$ and integrate with respect to $\tau$ from $s$ to $t+1$ to obtain

$$|\nabla \boldsymbol{u}_\epsilon(t+1)|^2 \leq |\nabla \boldsymbol{u}_\epsilon(s)|^2 + \frac{|\boldsymbol{f}|^2}{\nu}(t+1-s).$$

We then integrate the above inequality with respect to $s$ from $t$ to $t+1$ to get

$$|\nabla \boldsymbol{u}_\epsilon(t+1)|^2 \leq \int_t^{t+1} |\nabla \boldsymbol{u}_\epsilon(s)|^2 ds + \frac{|\boldsymbol{f}|^2}{2\nu}.$$

It follows from (29) that

$$|\nabla \boldsymbol{u}_\epsilon(t)|^2 \leq \frac{1}{\nu}\left(r_0^2 + \frac{|f|^2}{\alpha} + \frac{|f|^2}{2}\right) = R_1^2, \quad \forall\, t \geq t_0(B_{R_0}) + 1.$$

This means for $\boldsymbol{u}_{\epsilon 0} \in B_{R_0}$ and any $\tau \geq t_0(B_{R_0}) + 1$, we have $\boldsymbol{u}_\epsilon(\tau) \in B_{R_1} = \{\boldsymbol{u}_\epsilon(\tau) \in \hat{V};\ |\nabla \boldsymbol{u}_\epsilon(\tau)| \leq R_1\}$. That is,

$$W_\epsilon(\tau, 0; B_{R_0}) \subset B_{R_1}, \qquad \text{for } \tau \geq t_0(B_{R_0}) + 1. \tag{30}$$

From Corollary 3.2, it is easy to show

$$|\nabla \boldsymbol{u}_\epsilon(t)|^2 \leq |\nabla \boldsymbol{u}_\epsilon(\tau)|^2 e^{-\alpha(t-\tau)} + \rho_1^2\left(1 - e^{-\alpha(t-\tau)}\right), \quad \forall\, t > \tau \geq t_0(B_{R_0}) + 1,$$

where $\alpha = \nu\mu_1$ and $\rho_1^2 = |f|^2/\nu\alpha$. Again, both $\alpha$ and $\rho_1$ are independent of $\epsilon$. Hence, for any ball $B_{R_1}$, there exists a ball $B_{r_1}$ in $\hat{V}$ centered at origin with radius $r_1 > \rho_1$ ($R_1 > r_1$) such that

$$W_\epsilon(t, \tau; B_{R_1}) \subset B_{r_1}, \qquad \text{for } t \geq t_1(B_{R_0}) = t_0(B_{R_0}) + 1 + \frac{1}{\alpha}\ln\frac{R_1^2 - \rho_1^2}{r_1^2 - \rho_1^2}. \tag{31}$$

The ball $B_{r_1}$ is said to be **exponentially absorbing** and **invariant** under the action of the map $W_\epsilon(t, \tau; \cdot)$. Combining results in (30) and (31) and applying the semigroup property gives us

$$\begin{aligned} W_\epsilon(t, 0; B_{R_0}) &= W_\epsilon(t, \tau; W_\epsilon(\tau, 0; B_{R_0})), \quad \tau \geq t_0(B_{R_0}) + 1 \\ &= W_\epsilon(t, \tau; B_{R_1}) \subset B_{r_1}, \quad \forall\, t \geq t_1(B_{R_0}). \end{aligned}$$

That is, ball $B_{r_1}$ of radius $r_1$ in $\hat{V}$ will absorb any bounded set $B_{R_0}$ in $H$. Since the embedding of $\hat{V}$ in $H$ is compact, we deduce that $W_\epsilon(t, 0; \cdot)$ maps a bounded set in $H$ into a compact set in $H$. In addition, the operators $W_\epsilon(t, 0; \cdot)$ are **uniformly compact** for $t \geq t_1(B_{R_0})$. That is,

$$\bigcup_{t \geq t_1} W_\epsilon(t, 0; B_{R_0})$$

is relatively compact in $H$. Due to a theorem from Temam [16, Theorem 1.1], there exists a compact attractor $\Lambda_\epsilon$ which attracts every bounded sets in $H$. $\Lambda_\epsilon$ is the global attractor for

operators $W_\epsilon(t,0;\cdot)$ and it is also the $\omega$-limit set of the absorbing set $B_{r_1}$ (i.e. $\Lambda_\epsilon = \omega(B_{r_1})$). This means if we denote $W_\epsilon(t,0;B_{r_1}) = B_{r_1}(t)$ then

$$\Lambda_\epsilon = \bigcap_{\tau\geq 0} cl\left(\bigcup_{t\geq\tau} B_{r_1}(t)\right).$$

Note that the global attractor $\Lambda_\epsilon$ must be contained in the absorbing balls in $H$ and $\hat{V}$:

$$\Lambda_\epsilon \subseteq B_{r_0} \cap B_{r_1}.$$

Notice that all the above bounds are independent of $\epsilon$. In particular, for $\epsilon = 0$, we obtain existence of the compact attractor $\Lambda$ for the conventional Navier-Stokes equations. We now prove the following theorem:

**Theorem 6.1** *For $\epsilon \geq 0$, $W_\epsilon$ admits a compact attractor $\Lambda_\epsilon$ which attracts bounded sets of $H$ and is contained in the absorbing balls $B_{r_0} \cap B_{r_1}$, where $r_0$ and $r_1$ are independent of $\epsilon$. Moreover, $d_H(\Lambda_\epsilon, \Lambda) \to 0$ as $\epsilon \to 0$.*

**Proof:** We need the following result [4, Lemma 2.1]:

Let $X$ be a Banach space and $T(t)$, $t \geq 0$, a semigroup on $X$. Let $T_h(t)$ be an approximate semigroup (depending on a parameter $h > 0$) to the semigroup $T(t)$. For $\delta > 0$, let $\mathcal{N}(\mathcal{B},\delta)$ denote the $\delta$-neighborhood of a bounded set $\mathcal{B} \in X$ which is the union of open balls of radius $\delta$ centered on $\mathcal{B}$.

**Proposition 6.1** *Let $\mathcal{B} \in X$ be a bounded set and $\mathcal{U}_0, \mathcal{U}_1$ be two open sets such that $\mathcal{N}(\mathcal{B},d_0) \subset \mathcal{U}_0$, $\mathcal{N}(\mathcal{B},d_1) \subset \mathcal{U}_1$ for some $d_0, d_1 > 0$. If*

(i) *$\mathcal{B}$ attracts $\mathcal{U}_0$ under $T(t)$ and*

(ii) *$T_h(t)$ approximates $T(t)$ on $\mathcal{U}_1$ uniformly on compact sets of $[t_0,\infty)$, with $t_0 \geq 0$, then for any $\delta > 0$, there are $h_0 > 0$ and $\tau_0 > t_0$ such that, for $0 < h < h_0$, for $t \geq \tau_0$,*

$$T_h(t)\,(\mathcal{U}_0 \cap \mathcal{U}_1) \subset \mathcal{N}(\mathcal{B},\delta).$$

□

We will now prove the semicontinuity property for the global attractor $\Lambda_\epsilon$. This can be proven by showing the hypotheses of above proposition are satisfied by $W(t,0;\cdot)$ and $W_\epsilon(t,0;\cdot)$ for $\epsilon$ small enough. Clearly, it is sufficient to show that the $\delta$-neighborhood of attractor $\Lambda$ is an absorbing set and that $W_\epsilon(t,0;\cdot)$ approximates $W(t,0;\cdot)$ on $B_{R_2} = \{\boldsymbol{u}_\epsilon(0) \in \hat{V};\ |\nabla \boldsymbol{u}_\epsilon(0)| \leq R_2\}$ uniformly on compact sets of $[0,\infty)$. Let us prove this in two steps:

**First step:**

Let $\mathcal{N}(\Lambda,\delta)$ be the $\delta$-neighborhood of the attractor $\Lambda$. Since $\Lambda$ is a global attractor, for any bounded set $B_{R_0} = \{\boldsymbol{u}(0) \in H;\ |\boldsymbol{u}(0)| \leq R_0\} \subset H$

$$d_H(W(t,0;B_{R_0}),\Lambda) \longrightarrow 0 \qquad \text{as } t \to \infty.$$

Here $d_H(A,B)$ is the semidistance of two subsets $A, B$ of $H$:

$$d_H(A,B) = \sup_{x\in A}\ \inf_{y\in B}\ d(x,y).$$

Thus, there exists $\delta > 0$ and $t \geq t_\delta$ such that

$$d_H(W(t,0;B_{R_0}),\Lambda) \leq \frac{\delta}{2}, \quad \text{for } t \geq t_\delta.$$

This implies

$$W(t,0;B_{R_0}) \subset \mathcal{N}(\Lambda,\delta), \quad \text{for } t \geq t_\delta.$$

This shows that $\mathcal{N}(\Lambda,\delta)$ is an absorbing set.

**Second step:**

We want to show $W_\epsilon(t,0;\cdot)$ approximates $W(t,0;\cdot)$ on $B_{R_2}$ uniformly on compact sets of $[0,\infty)$. By subtracting (13) from (12), we obtain for $\boldsymbol{w} = \boldsymbol{u}_\epsilon - \boldsymbol{u}$

$$\boldsymbol{w}' + \epsilon A\boldsymbol{u}_\epsilon + \nu A_1 \boldsymbol{w} + B(\boldsymbol{u}_\epsilon,\boldsymbol{u}_\epsilon) - \hat{B}(\boldsymbol{u},\boldsymbol{u}) = 0.$$

Taking the inner product with $\boldsymbol{w}$, we get

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{w}|^2 + \epsilon(A\boldsymbol{u}_\epsilon,\boldsymbol{w}) + \nu|\nabla\boldsymbol{w}|^2 = b(\boldsymbol{w},\boldsymbol{w},\boldsymbol{u}_\epsilon). \tag{32}$$

Since $D(A^{1/2}) = V = D(A_1)$, the second term of left hand side of equation (32) can be written as

$$\begin{aligned}\epsilon(A\boldsymbol{u}_\epsilon, \boldsymbol{w}) &= \epsilon(A\boldsymbol{u}_\epsilon, \boldsymbol{u}_\epsilon) - \epsilon(A\boldsymbol{u}_\epsilon, \boldsymbol{u})\\ &= \epsilon|A^{1/2}\boldsymbol{u}_\epsilon|^2 - \epsilon(A^{1/2}\boldsymbol{u}_\epsilon, A^{1/2}\boldsymbol{u})\\ &= \epsilon|A^{1/2}\boldsymbol{u}_\epsilon|^2 - \epsilon(A^{1/2}\boldsymbol{u}_\epsilon, A_1\boldsymbol{u}).\end{aligned}$$

This leads to

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{w}|^2 + \epsilon|A^{1/2}\boldsymbol{u}_\epsilon|^2 + \nu|\nabla\boldsymbol{w}|^2 = \epsilon(A^{1/2}\boldsymbol{u}_\epsilon, A_1\boldsymbol{u}) + b(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{u}_\epsilon).$$

By applying Young's inequality, we have

$$\begin{aligned}\epsilon(A^{1/2}\boldsymbol{u}_\epsilon, A_1\boldsymbol{u}) &\le \epsilon|A^{1/2}\boldsymbol{u}_\epsilon|\,|A_1\boldsymbol{u}|\\ &\le \frac{\epsilon}{2}|A^{1/2}\boldsymbol{u}_\epsilon|^2 + \frac{\epsilon}{2}|A_1\boldsymbol{u}|^2.\end{aligned}$$

The trilinear term can be estimated as

$$\begin{aligned}|b(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{u}_\epsilon)| &\le c_1|\boldsymbol{w}|^{1/2}|\nabla\boldsymbol{w}|^{1/2}|\boldsymbol{w}|^{1/2}|\nabla\boldsymbol{w}|^{1/2}|\nabla\boldsymbol{u}_\epsilon|\\ &\le c_1|\boldsymbol{w}|\,|\nabla\boldsymbol{w}|\,|\nabla\boldsymbol{u}_\epsilon|\\ &\le \frac{\nu}{2}|\nabla\boldsymbol{w}|^2 + \frac{c_1^2}{2\nu}|\boldsymbol{w}|^2|\nabla\boldsymbol{u}_\epsilon|^2.\end{aligned}$$

By combining all of the above estimates in (32), we get

$$\frac{d}{dt}|\boldsymbol{w}|^2 + \epsilon|A^{1/2}\boldsymbol{u}_\epsilon|^2 + \nu|\nabla\boldsymbol{w}|^2 \le \epsilon|A_1\boldsymbol{u}|^2 + \frac{c_1^2}{\nu}|\boldsymbol{w}|^2|\nabla\boldsymbol{w}|^2.$$

We can drop the positive terms $\epsilon|A^{1/2}\boldsymbol{u}_\epsilon|^2$ and $\nu|\nabla\boldsymbol{w}|^2$ to obtain the following differential inequality

$$\frac{d}{dt}|\boldsymbol{w}|^2 \le \epsilon|A_1\boldsymbol{u}|^2 + \frac{c_1^2}{\nu}|\boldsymbol{w}|^2|\nabla\boldsymbol{u}_\epsilon|^2. \tag{33}$$

Using Corollary 3.2 we get,

$$|\nabla\boldsymbol{u}_\epsilon(t)|^2 \le |\nabla\boldsymbol{u}_\epsilon(0)|^2 e^{-\alpha t} + \rho_1^2(1 - e^{-\alpha t}).$$

This implies that any ball $B_{R_2} = \{\boldsymbol{u}_\epsilon(0) \in \hat{V};\ |\nabla \boldsymbol{u}_\epsilon(0)| \leq R_2\}$ in $\hat{V}$ with radius $R_2 > \rho_1$ will satisfy

$$W_\epsilon(t, 0; B_{R_2}) \subset B_{R_2}, \qquad \text{for } t \geq 0.$$

This means if $\boldsymbol{u}_\epsilon(0) \in B_{R_2}$, then $W_\epsilon(t, 0; \boldsymbol{u}_\epsilon(0))$ is defined and belongs to $B_{R_2}$ for $t \geq 0$. The ball $B_{R_2}$ is therefore **invariant** under the map $W_\epsilon(t, 0; \cdot)$. That is , we have

$$|\nabla \boldsymbol{u}_\epsilon(t)| \leq R_2, \quad t \geq 0, \quad \text{for } \boldsymbol{u}_\epsilon(0) \in B_{R_2}. \tag{34}$$

Let $\boldsymbol{u}_\epsilon(0) \in B_{R_2}$ then it follows from (33) that

$$\frac{d}{dt}|\boldsymbol{w}|^2 \leq \epsilon |A_1 \boldsymbol{u}|^2 + \frac{c_1^2 R_2^2}{\nu}|\boldsymbol{w}|^2, \quad t \geq 0. \tag{35}$$

From the standard Gronwall Lemma, (35) gives

$$\begin{aligned}
|\boldsymbol{w}(t)|^2 \ &\leq |\boldsymbol{w}(0)|^2 \exp\left(\int_0^t \frac{c_1^2 R_2^2}{\nu} d\tau\right) \\
&\quad + \epsilon \int_0^t |A_1 \boldsymbol{u}(s)|^2 \exp\left(\int_s^t \frac{c_1^2 R_2^2}{\nu} d\tau\right) ds \\
&= \epsilon \int_0^t |A_1 \boldsymbol{u}(s)|^2 \exp\left[\frac{c_1^2 R_2^2 (t-s)}{\nu}\right] ds,
\end{aligned}$$

where $\boldsymbol{w}(0) = \boldsymbol{u}_\epsilon(0) - \boldsymbol{u}(0) = 0$.

Let us now consider $t \in [0, T]$ such that $0 \leq s \leq t \leq T$. This gives

$$|\boldsymbol{w}(t)|^2 \leq \epsilon \ \exp\left(\frac{c_1^2 R_2^2}{\nu} T\right) \int_0^t |A_1 \boldsymbol{u}(s)|^2 ds.$$

By applying (ii) of Corollary 3.2 (with $\epsilon = 0$), we obtain

$$|\boldsymbol{w}(t)|^2 \leq \frac{\epsilon}{\nu} \ \exp\left(\frac{c_1^2 R_2^2}{\nu} T\right) \left(R_2^2 + \frac{|\boldsymbol{f}|^2 T}{\nu}\right), \quad t \in [0, T].$$

This means for $\boldsymbol{u}_\epsilon(0) \in B_{R_2}$, then

$$|W_\epsilon(t, 0; \boldsymbol{u}_\epsilon(0)) - W(t, 0; \boldsymbol{u}_\epsilon(0))|_H^2 \leq g(\epsilon, \nu, R_2, \boldsymbol{f}, T),$$

with

$$g(\epsilon, \nu, R_2, \boldsymbol{f}, T) = \epsilon \left(\frac{R_2^2}{\nu} + \frac{|\boldsymbol{f}|^2 T}{\nu^2}\right) \exp\left(\frac{c_1^2 R_2^2}{\nu} T\right)$$

and

$$\lim_{\epsilon \to 0} g(\epsilon, \nu, R_2, f, T) = 0.$$

Hence $W_\epsilon(t)$ **approximates** $W(t)$ on $B_{R_2}$ uniformly on $[0, T]$. According to the Proposition 6.1, then for any $\delta > 0$, there are $\epsilon_0 > 0$ and $\tau_0 > 0$ such that

$$W_\epsilon(t)\,(B_{R_0} \cap B_{R_2}) \subset \mathcal{N}(\Lambda, \delta), \quad \text{for } 0 < \epsilon < \epsilon_0,\ t \geq \tau_0.$$

Since the attractor $\Lambda_\epsilon$ is contained in $B_{R_0} \cap B_{R_2}$, we have

$$W_\epsilon(t)(\Lambda_\epsilon) \subset \mathcal{N}(\Lambda, \delta), \quad \text{for } 0 < \epsilon < \epsilon_0,\ t \geq \tau_0.$$

Since $\Lambda_\epsilon$ is an invariant set, we deduce that

$$\Lambda_\epsilon \subset \mathcal{N}(\Lambda, \delta), \quad \text{for } 0 < \epsilon < \epsilon_0,\ t \geq \tau_0.$$

Since $\delta$ is arbitrary, we obtain the upper semicontinuity of $\Lambda_\epsilon$ at $\epsilon = 0$:

$$d_H(\Lambda_\epsilon, \Lambda) \longrightarrow 0, \quad \text{as } \epsilon \to 0.$$

$\square$

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